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What is a covariant derivative?

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Abstract. *A slight change of emphasis, from geometry towards local symmetry, provides a better framework for the formulation of gauge theories of the gravitational type. General Relativity no longer requires any improvisation, whether the local algebra is based on Poincaré or on De Sitter. Application to the graded super Minkowski manifold, with local algebra based on the super Poincaré algebra, leads to an action principle.* **The torsion constraints are obtained by straightforward variation of the action.** *The comparison with component supergravity has not yet been completed, but preliminary indications strongly indicate a close relationship. Local symmetry of the action principle is maintained, the structure is field independent, and the torsion constraints also preserve the local symmetry algebra. The coupling to matter shows that these constraints are necessary for the exclusion of ghosts from the matter sector.*

1. INTRODUCTION

The concept of «gauge theories» has become dominant in physics, but is it well defined? Before attempting to answer this question, let us think about whether it is worth the trouble. Certainly, one may consider the many brilliant successes already obtained in the field as proof positive that the foundations are solid, that the path to discovery matters but little after the fact. With this we do not wish to quarrel; instead, we want to emphasize that, actually, success has been anything but complete. The purpose to which we shall discuss the structure of gauge theories is constructive, and the aim is to attack real problems. In order to attempt to engage the interest of the reader it would be best to lay

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before him, immediately, some concrete results, to show that a review of the methodology of gauge theories can bring tangible dividends. But it is necessary first to define the problem.

The greatest success story of recent times is without any doubt the prediction of the existence and the masses of the electroweak mesons [1]. One of the main ideas behind this theory was the requirement of internal consistency, more precisely, renormalizability [2]. It embodies a principle that appears to be of great importance: renormalizability is much enhanced by symmetry, exact or very carefully broken. This same principle animated, to a great extent, early interest in supersymmetry. There is no doubt that supersymmetric theories are more finite [3], but what counts is renormalizability, not finiteness. Great hope was attached to the prospect that super-gravity might provide a consistent quantum gravity, but here success has been only partial [4]; in other words, as long as that situation remains, negligible. And yet, it seems to us, it is too early to write off supergravity.

Supergravity was never given a real chance to realize its potential. The high hopes were based on the exploitation of symmetry, but the symmetry of supergravity must first be fully implemented. As was first predicted by Dirac, renormalization of electrodynamic was achieved only within a formulation with *manifest* Poincaré symmetry. In quantum mechanics it is well known that a formal symmetry of the hamiltonian does not imply degeneracy of the spectrum; one knows well the relevance of domains and integrability for a symmetry to be of consequence. In quantum field theory such insight is lacking, but one would surely do well to be careful when trying to draw inference from the «symmetry» of supergravity. First of all, integrability, so important in the quantum mechanical context, is certainly not provable; one has to be content with an invariance algebra rather than an invariance group. But worse than that, the «algebra» is not defined, for it does not have a structure independently of its realizations. (The «structure tensor» depends on the dynamical fields). The search for a superspace formulation of supergravity was undertaken (or could at least be justified this way) to improve the theory in this respect. It is fully recognized, after all, that the vaunted properties of super Yang-Mills theories cannot be established without a superspace formulation that is free of constraints.

The first superspace formulation of supergravity was achieved by Wess and Zumino [5], it was a real tour de force of hard work and brilliant improvisation. But their theory does not satisfy the demands that one must put on it if one would advance the hopes of a renormalizable version of supergravity. There is a stage, before the imposition of constraints, where one can speak of a local symmetry algebra (the good one?). But at this level one does not have an action principle. The constraints are no doubt important, but they are not derived from an action. They have to be introduced from outside, and they leave us with a theory that shares the drawbacks of the original component formulation. Another monumental effort, by Arnowitt and Nath [6], showed a lot of promise; it was eventually abandoned without having been fully developed, for reasons that are not too clear. Finally, Ogievetsky and Sokatchev have developed a very original approach [7], perhaps the most aesthetic one so far, marred, nevertheless, by the same imperfections as the theory of Wess and Zumino [5].

In view of this state of affairs, the question of renormalizability of supergravity must be regarded as being still open. It will probably remain open at least until one shall have a formulation in which all the constraints are derived from an action principle that is invariant under an appropriate local algebra. It appears as if our investigation into the structure of gauge theories has led us to such a formulation. This paper presents an action principle, invariant under a local super Poincaré algebra (field independent structure!), from which a weakened form of the Wess-Zumino torsion constraints follows *automatically*. It admits flat superspace as an exact solution. A preliminary investigation of matter couplings (in a formulation in which all constraints come from the action principle) gives strong support for our belief that this theory is the required extension of supergravity, but a proof is not yet at hand.

Gauge theory is a framework for the construction of dynamical field theories, especially new theories. Many attempts have been made to use it effectively, to further the understanding of gravity, to invent conformal gravity, to formulate supergravity, to limit ourselves to the gravitational context. Still, it is by no means agreed by one and all that General Relativity is a gauge theory in the sense that Yang-Mills theories are gauge theories. Among the partisans of the view that Einstein's theory is a gauge theory there is no general agreement about the correct choice of gauge group. This being the case, it is clear that General Relativity cannot (yet) serve as a paradigm for the invention of new theories. Too much play is left for improvisation. Section 2 of this paper attempts to improve this situation. Here we shall review first, the applications that shall be carried out in the later sections, and subsequently, the general framework that has been formulated in Section 2.

Section 3 deals with ordinary gravity. The gauge algebra is local Poincaré; attempts to use only the Lorentz algebra do not succeed, for reasons that are discussed. The framework of Section 2 is applied, here as in the other cases, without the slightest improvisation. **General Relativity is, in our opinion, the gauge theory par excellence**. The approach used here completely solves (avoids) the perennial confusion between vierbein and connection coefficients. Section 4 is entitled De Sitter gravity, for the gauge algebra is local De Sitter. The algorithm works just as well in this case, and the result is again General Relativity.

Differences between Poincaré gravity and De Sitter gravity appear only in the case of coupling to spinors.

Section 5 applies exactly the same procedure to the case of the Poincaré super algebra. Superspace and «tangent» space are, as usual, of graded dimension 4 + 4. In line with the general ideas explained in Section 2, we allow the super translations, and not just the Lorentz transformations, to act nontrivially on tangent space. This is crucial because the special limiting case in which they do *not* act is characterized by an ambiguity that obfuscates the correct choice of the action. The invariantly contracted curvature tensor contains only one half of the 64 achtbein coefficients, and one half of the connection coefficients enter only algebraically. This explains why the torsion constraints appear naturally and automatically, as Euler-Lagrange equations. These constraints do not violate local supersymmetry; they are slightly weaker than those of Wess and Zumino [5]. In Section 6 we establish the fixed point of flat space supersymmetry. We have not yet completed the reduction of the action to component language. Our belief that the theory is an extension of supergravity is based mostly on the developments of the last sections.

Section 7 deals with the minimal coupling of supergravity to matter. Our description of super-symmetric matter differs from the traditional one and needs to be explained. The usual approach, as developed by Wess and Zumino [8], and by Salam and Strathdee [9], deals with chiral superfields. This method has proved its mettle in many successful applications, including supergravity and super Yang-Mills [10, 11]. Nevertheless, we still harbor some reservations about it. First of all, the superfield action of the Wess-Zumino multiplet is an algebraic expression, containing no derivatives of the superfield. This is explained by the fact that the chiral superfield satisfies differential constraints, the space-time derivatives are hidden in the expressions for the components, which means, of course, that the action is to be varied subject to these constraints. In other words, this approach to superspace field theory is very different from the traditional methods developed for ordinary field theories. In fact, differential constraints are banned there, for very good reasons. Our point of view is consistent with that advanced above, in connection with supergravity. It must be admitted, however, that an aesthetic prejudice constitutes much of our motivation to avoid the introduction of *a priori* constraints of any type, and chirality constraints in particular [12].

The free field equations for the unconstrained, scalar superfield can be derived from an action that looks very much like that of an ordinary scalar field. The field equations include all the constraints, which is satisfying but also dangerous. Interactions, especially those that involve derivatives, tend to destroy the constraints and thus increase the number of degrees of freedom. This threatens he unitarity of the theory, which is why one may justificably refer to the new tates as ghosts. One is especially vulnerable to the appearance of ghosts in the ;ase of interest, for the minimal supergravitational interaction contains just as nany derivatives as the free action. It is a remarkable fact that the torsion con- traints derived from variation of the super-gravity connection, that remain ınchanged by the coupling to the scalar matter superfield, seem to be exactly vhat is needed to avoid ghosts in the matter sector, though they are weaker than he constraints of Wess and Zumino [5]. This circumstance makes us feel rather ɔositive about prospects for developing a sensible theory, and that is why we ɔelieve that supergravity is included correctly.

It remains to explain what is new in the general framework that has been set up in Section 2, from which everything follows. At first sight, not much. We choose a Lie algebra g, of finite dimension, a manifold M, and introduce the local algebra lg, as usual. We insist that g acts on M; if this action is trivial, then one has a gauge theory of the Yang-Mills type, a very special case. We are mostly interested in the opposite case, when the action of g is effective, by which we mean that the vector fields of g span the tangent space of M at each point In that case the action of lg on the space M (but not on the fields) reduces to the infinitesimal diffeomorphisms, and *diff*(M) **thus appears through a natural homomorphism**. This unambiguous appearance of *diff*(M) is not completely new, but it is as important as it is unusual. Notice that *diff*(M) invariance comes about automatically, it is not a postulate; we are not partisans of a «pure geo- metrical» point of view. This independence of received geometrical notions is particularly in evidence when it comes to the next step, which is the definition of the covariant derivative. We insist that all fields of interest be g-modules; it follows that the covariant derivative operates between g-modules. This principle is sufficient to determine the transformation properties of connection, vielbein coefficients, curvature and torsion under lg, which constitutes an important departure from those approaches in which the vielbein is introduced as an extra ad hoc element. It also leads to the two identities (2.30) and (2.31). These iden- tities are very important, as we try to make clear especially in the application to Poincaré gravity. Other ambiguities remain: the choice of g, of M, and of tangent space. The scalar curvature serves as invariant Lagrangian in some of the most important cases, but not always. There is no general principle that can tell us the correct choice of the action.

The matter of choosing a tangent space deserves some comment. The usual, four-dimensional tangent space used for ordinary gravity is natural from every point of view, so this example is not much help in deciding what it is that matters most. We do insist that tangent space must be a g-module, and such as to allow for the existence of an invariant curvature scalar (which rules out trivial g-modules).

This may lead to difficulties with dimension, unless one is willing to accept that the matrix of vielbein coefficients may be rectangular (not square). Thus, in conformal gravity, the dimension of tangent space exceeds that of the manifold, since the lowest dimension of a real, non-trivial representation of the conformal group is six. The reverse case is even more interesting. In supergravity it has been customary to use an eight-dimensional super manifold and an eight-dimensional tangent space. That is how we start out also, but the analysis ends up with a four-dimensional spinorial tangent space. It turns out that this deficiency in the dimension of tangent space is responsable for the torsion constraints. Some preliminary reflections on this idea are offered in Section 8.

2. GENERAL STRUCTURE

Let M be a differentiable manifold and g a Lie algebra. Let (ℓ_A), $A = 1, \ldots, n$ be a basis for g and suppose that g acts in M by differentiable vector fields («orbital» action).

$$\ell_A \to M_A, \quad A = 1, \ldots, n.$$

If V is any vector space, let $\mathscr{F}(M, V)$ denote the space of differentiable functions from M to V. If V is a g-module, in which g acts by matrices

$$\ell_A \to S_A, \quad A = 1, \ldots, n,$$

then g acts in $\mathscr{F}(M, V)$ by the operators («orbital» plus «spin» action)

$$\ell_A \to L_A = M_A + S_A;$$

hence $\mathscr{F}(M, V)$ is a g-module.

Let V be any g-module, and let V_g be g considered only as a vector space, on which g acts by the adjoint representation. The space $lg(V)$ of operators in $\mathscr{F}(M, V)$ of the form

$$\Lambda \cdot L = \Lambda^A L_A, \quad \Lambda \in \mathscr{F}(M, V_g)$$

is a Lie algebra. If V is a faithful g-module, then the structure relations

$$[\Lambda' \cdot L, \Lambda \cdot L] = \Lambda'' \cdot L$$

are independent of V and take the form (C is the structure tensor of g)

$$\Lambda''^C = \Lambda'^A \Lambda^B C_{AB}{}^C + \xi' \Lambda^C - \xi \Lambda'^C, \tag{2.1}$$

$$\xi = \Lambda^A M_A, \quad \xi' \equiv \Lambda'^A M_A.$$

DEFINITION (2.2). *The local algebra* lg *is the space* $\mathscr{F}(M, V_g)$ *with the structure* *of Lie algebra defined by Eq.* (2.1) [13].

If V is a g-module, faithfull or not, then $\mathscr{F}(M, V)$ is an $\mathrm{l}g$-module, and the action of g in this space defines a homomorphism from g to some sub-quotient $\mathrm{l}g(V)$. If V is a trivial g-module, then $\mathrm{l}g(V)$ is a subalgebra of the Lie algebra *diff*(M) of differentiable vector fields on M. We shall here suppose that g acts effectively in M; that is, that (M_A), $A = 1, \ldots, n$, evaluated at $x \in M$ spans the tangent space at x. Then $\mathrm{l}g(V) = diff(M)$ whenever g acts trivially on V. To be precise, one has the homomorphism

$$(2.3) \qquad \pi : \mathrm{l}g \to diff(M), \quad \Lambda \mapsto \xi = \Lambda \cdot M.$$

There is a good reason for insisting on this intimate connection between the local algebra and the diffeomorphism algebra, for it makes the interpretation of General Relativity as a gauge theory completely natural. Here we are in emphatic disagreement with much of the literature, and especially with proposals to look at $\mathrm{l}g$ and *diff*(M) as independent ingredients of the theory.

Later we shall have occasion to introduce the subalgebra $\mathrm{sl}g$ defined by

$$(2.4) \qquad M_A \Lambda^A = 0.$$

One easily verifies that this does in fact define a subalgebra of $\mathrm{l}g$

The local algebra also acts on world tensor fields over M. Let M, V be as above, and let $\mathscr{T}(M, V)$ be the space of differentiable tensor fields over M, valued in V. This is an $\mathrm{l}g$-module with the natural action

$$(2.5) \qquad \Lambda \to \mathscr{L}_\Lambda = \mathscr{L}(\xi) + \Lambda \cdot S.$$

Here ξ is the vector field defined in (2.1), and $\mathscr{L}(\xi)$ is the ordinary Lie derivative associated with ξ. The operator $\mathscr{L}_\Lambda$ in $\mathscr{T}(M, V)$ will be called the Lie derivative associated with Λ. If V is a trivial g-module, then $\mathscr{T}(M, V)$ reduces to the usual tensorial *diff*(M) module.

For fixed g and M we now turn to the problem of defining a covariant derivative. It seems natural to interpret the local algebra $\mathrm{l}g$ as a bundle over *diff*(M), with bundle projection π defined by Eq. (2.3). Avoiding the difficult problem of introducing topologies on these spaces, we propose

DEFINITION (2.6). *A "connection one-form" is a section of the homomorphism (2.3) that is, a map* $\Gamma: diff(M) \to \mathrm{l}g$ *such that* $\pi \circ \Gamma$ *is the identity map*:

$$(2.7) \qquad \begin{aligned} &diff(M) \ni \xi \to \Gamma(\xi) \in \mathrm{l}g, \\ &(\pi \circ \Gamma)(\xi) = \xi \end{aligned}$$

This notion will be relevant and useful, but we shall nevertheless reserve the name «connection» for a slightly different object. Note that (2.7) is a nontrivial constraint on the coefficients; see Eq. (2.31) below.

To define a covariant derivative we must fix, besides g and M, a «tangent space», so called for quaint historical reasons [14]. From now on $\widetilde{V}$ will denote a fixed, finite-dimensional g-module. We insist that this «tangent space» must be a g-module.

DEFINITION (2.8). *A «connection» is a differentiable function* $M \to \widetilde{V} \otimes V_g$:

$$\phi = (\phi_\alpha{}^A), \quad \alpha = 1, \ldots, d; \quad A = 1, \ldots, n,$$

where d is the dimension of $\widetilde{V}$.

A connection defines, for each g-module V, a map from the space of functions with values in V to the space of functions with values in $\widetilde{V} \otimes V$,

$$(2.9) \qquad Q(\phi) : \mathscr{F}(M, V) \to \mathscr{F}(M, \widetilde{V} \otimes V).$$

It is determined by the d operators of covariant differentiation

$$(2.10) \qquad Q_\alpha = \phi_\alpha{}^A L_A, \quad \alpha = 1, \ldots, d.$$

Thus Q operates between g-modules, but it is not a module map.

The space $\mathscr{F}(M, \widetilde{V} \otimes V_g)$ of connections is turned into an Ig-module by the homomorphism $\Lambda \to \delta_\Lambda$, where δ_Λ acts in $\mathscr{F}(M, \widetilde{V} \otimes V_g)$ according to the following rule. To first order in ϵ,

$$(1 + \epsilon \mathscr{L}_\Lambda) \cdot Q(\phi) = Q(\phi + \epsilon\delta_\Lambda \phi) \cdot (1 + \epsilon \mathscr{L}_\Lambda)$$

Thus

$$(2.11) \qquad Q(\delta_\Lambda \phi) = \mathscr{L}_\Lambda \cdot Q(\phi) - Q(\phi) \cdot \mathscr{L}_\Lambda .$$

DEFINITION (2.12). *A covariant derivative is a space $\mathscr{F}(M, \widetilde{V} \otimes V_g)$ of connections, or the associated space of maps given by (2.9-10), with the structure of* Ig*-module given by (2.11).*

The Lie algebra g acts in $\widetilde{V}$ by matrices:

$$e_A \to \widetilde{S}_A .$$

Evidently, when the operator $\mathscr{L}_\Lambda$ is applied after $Q(\phi)$, as in (2.11), $S_A + \widetilde{S}_A$ replaces S_A in Eq. (2.5). A short calculation gives

$$(2.13) \qquad (\delta_\Lambda \phi)_\alpha{}^A = (\mathscr{L}_\Lambda \phi)_\alpha{}^A \quad e_\alpha \Lambda^A ,$$

in which e_α is the vector field

(2.14) $$e_\alpha = \phi_\alpha{}^A M_A .$$

The action (2.11) induces

(2.15) $$(\delta_\Lambda e)_\alpha = (\mathscr{L}_\Lambda e)_\alpha ,$$

so that e transforms as a $\widetilde{V}$-valued world vector field, while ϕ does not transform as a $\widetilde{V} \otimes V_g$-valued function, witness the last term in (2.13).

If we want to make contact with the concept introduced in Definition (2.6), then we must assume that ϕ factorizes. Let x_μ be local coordinates on M and define vielbein coefficients by

$$e_\alpha = e_\alpha{}^\mu\, \partial_\mu , \quad \partial_\mu = \partial/\partial x_\mu .$$

For the present purpose only, suppose that ϕ has the representation

(2.16) $$\phi_\alpha{}^A = e_\alpha{}^\mu\, \Gamma_\mu{}^A .$$

We stress that this need not be assumed in general, especially since the matrix $(e_\alpha{}^\mu)$ may be rectangular, in which case the factorization may be impossible or ambiguous. If (2.16) holds, then

(2.17) $$Q_\alpha = e_\alpha{}^\mu\, D_\mu ,$$

with

(2.18) $$D_\mu = \partial_\mu + \Gamma_\mu{}^A\, S_A .$$

If the matrix $(e_\alpha{}^\mu)$ is invertible, then the coefficients $\Gamma_\mu{}^A$ define a connection one form, $\Gamma(\xi) = \xi^\mu\, \Gamma_\mu{}^A\, \ell_A$, with the usual transformation law

(2.19) $$(\delta_\Lambda \Gamma)_\mu{}^A = (\mathscr{L}_\Lambda \Gamma)_\mu{}^A - \partial_\mu \Lambda^A .$$

Our reasons for emphasizing the connection as defined in (2.8), instead of the connection one-form defined in (2 6) is that this is the object that gauge theories are made of To limit ones attention to D_μ from the outset causes much trouble. This is especially true if the vielbein matrix $(e_\alpha{}^\mu)$ is not invertible or if the factorization (2 16) is not possible.

Note that iteration gives

(2.20) $$(Q \cdot Q)_{\alpha\beta} = Q_\alpha\, Q_\beta + \widetilde{\phi}_{\alpha\beta}{}^\gamma\, Q_\gamma$$

with

(2.21) $$\widetilde{\phi}_{\alpha\beta}{}^\gamma = \phi_\alpha{}^A\, (\widetilde{S}_A)_\beta{}^\gamma .$$

The practice of writing $Q_\alpha\, Q_\beta$ when $(Q \cdot Q)_{\alpha\beta}$ is meant can be misleading.

DEFINITION (2.22). *Torsion and curvature are defined by*

$$(Q \cdot Q)_{\alpha\beta} \mp (\alpha, \beta) = t_{\alpha\beta}{}^{\gamma} Q_{\gamma} + R_{\alpha\beta}{}^{A} S_{A} \,.$$

where the upper (lower) sign applies when ϕ is a Bose (Fermi) field.

The explicit expressions are

$$t_{\alpha\beta}{}^{\gamma} = c_{\alpha\beta}{}^{\gamma} + \tilde{\phi}_{\alpha\beta}{}^{\gamma} \mp \tilde{\phi}_{\beta\alpha}{}^{\gamma} , \tag{2.23}$$

$$R_{\alpha\beta}{}^{C} = \phi_{\alpha}{}^{A} \phi_{\beta}{}^{B} C_{AB}{}^{C} + (e_{\alpha} \phi_{\beta}{}^{C} \mp \alpha, \beta) - c_{\alpha\beta}{}^{\gamma} \phi_{\gamma}{}^{C} . \tag{2.24}$$

where the coefficients ($c_{\alpha\beta}{}^{\gamma}$) are defined only to the extent that

$$[e_{\alpha}, e_{\beta}] = c_{\alpha\beta}{}^{\gamma} e_{\gamma} \,. \tag{2.25}$$

The operators (2.18), when they exist, give rise to the curvature two-form ω:

$$[D_{\mu}, D_{\nu}] = \omega_{\mu\nu}{}^{A} S_{A} \,.$$

with

$$\omega_{\mu\nu}{}^{A} = \partial_{\mu} \Gamma_{\nu}{}^{A} - (\mu, \nu) + \Gamma_{\mu}{}^{B} \Gamma_{\nu}{}^{C} C_{BC}{}^{A} \,.$$

The formula

$$R_{\alpha\beta}{}^{A} = e_{\alpha}{}^{\mu}\, e_{\beta}{}^{\nu}\, \omega_{\mu\nu}{}^{A} \,. \tag{2.26}$$

relates ω to the curvature.

Invariant dynamics becomes possible if there is an invariant action, the first requisite for which is a curvature scalar: that is, any scalar function constructed from the curvature. The simplest case is the following. If the g-module $\tilde{V}$ has an invariant metric η, then a scalar field can be constructed by contraction of the curvature:

$$R = R_{\alpha\beta}{}^{A}\, \tilde{S}_{A}{}^{\alpha\beta} , \tag{2.27}$$

where

$$\tilde{S}_{A}{}^{\alpha\beta} = \eta^{\alpha\gamma} (\tilde{S}_{A})_{\gamma}{}^{\beta} . \tag{2.28}$$

Note that $R_{\alpha\beta}$ is antisymmetric if ϕ is a Bose field. For R to be non-zero we need an antisymmetric $\tilde{S}_{A}{}^{\alpha\beta}$ and consequently a symmetric η. If ϕ is a Fermi field, then $R_{\alpha\beta}$ is symmetric and η must be antisymmetric. The metric

$$g^{\mu\nu} = e_{\alpha}{}^{\mu}\, e_{\beta}{}^{\nu}\, \eta^{\alpha\beta} \tag{2.29}$$

is non-zero under the same conditions. The square root of the determinant of the metric, if it is defined, gives us the density needed to construct an invariant action integral.

Remark. It is an immediate consequence of the definitions that

(2.30) $$R_{\alpha\beta}{}^{A}\, M_A = 0.$$

holds identically. In the case that the factorization (2.16) is possible (which we do not assume in general), and if in addition the matrix of vielbein coefficients is invertible, then we have another identity,

(2.31) $$\Gamma_\mu{}^{A}\, M_A = \partial_\mu .$$

This is nothing but Eq. (2.7), the condition for Γ to be a section of the homomorphism $\pi : lg \to diff(M)$.

3. GAUGING GENERAL RELATIVITY

There has been a lot of discussion about the correct choice of gauge group, and even about the appropriate tangent space to use in General Relativity, so we consider several possibilities. The manifold is R^4.

(i) Take $\widetilde{V} = R^4$, η the Lorentzian metric and $g = so(\eta)$, the homogeneous Lorentz algebra [15]. By the normal connection between spin and statistics the field ϕ is a Bose field so that the contracted curvature (2.27) and the metric (2.29) do not vanish identically It is therefore natural to take the action density to be eR, where e is the determinant of $(e_\mu{}^{\alpha})$. It is possible to express this action density in terms of the metric. Since the local algebra acts on the metric by the ordinary Lie derivative, $\delta_\Lambda g = \mathscr{L}(\xi)\, g$, only the diffeomorphism algebra acts effectively in pure gravity. This remains true when matter couplings are introduced, provided that no spinor fields appear, for all tensor valued functions can be converted to world tensor fields with the help of the vierbein coefficients, and then δ_Λ always coincides with the Lie derivative, $\mathscr{L}_\Lambda = \mathscr{L}(\xi)$ with $\xi = \Lambda \cdot M$.

It will be useful to work out some of the details. A 4-by-4 matrix λ belongs to $so(\eta)$ iff $\lambda_{\alpha\beta} = \lambda_\alpha{}^{\gamma}\, \eta_{\gamma\beta}$ is antisymmetric We use the basis $(\ell_{\alpha\beta})$, $\alpha < \beta$, in which

(3.1) $$(\widetilde{S}_{\alpha\beta})_\gamma{}^{\delta} = \eta_{\alpha\gamma}\, \delta_\beta{}^{\delta} - (\alpha, \beta), \quad \widetilde{\phi}_\alpha{}^{\beta\gamma} = \phi_\alpha{}^{\beta\gamma}$$

then the curvature tensor, Eq. (2.24)

(3.2) $$R_{\alpha\beta}{}^{\gamma\delta} = \phi_\alpha{}^{\gamma\epsilon}\, \phi_{\beta\epsilon}{}^{\delta} + e_\alpha\, \phi_\beta{}^{\gamma\delta} - (\alpha, \beta) - c_{\alpha\beta}{}^{\epsilon}\, \phi_\epsilon{}^{\gamma\delta},$$

and the contracted curvature is $R_{\alpha\beta}{}^{\alpha\beta}$. In this case the matrix of vierbein coefficients is square, so the factorization hypothesis (2.16) can be made without essential loss of generality. In terms of the connection one form Γ, defined in Eq. (2.16), and the quantity

$$\tau_{\alpha\beta}{}^{\mu\nu} = e_\alpha{}^\mu e_\beta{}^\nu - (\alpha, \beta), \tag{3.3}$$

it is

$$R = \tau_{\alpha\beta}{}^{\mu\nu} (\partial_\mu \Gamma_\nu{}^{\alpha\beta} + \Gamma_\mu{}^{\alpha\gamma} \Gamma_{\nu\gamma}{}^\beta) . \tag{3.4}$$

[**Notation**: Late Greek letters $\mu, \nu, \ldots$ are used exclusively as world tensor indices. Indices are moved and converted with the aid of the metrics g and η and the vierbein coefficients. The operator D_ν was defined in (2.17-18)]

If the coefficients of Γ and of e were independent of each other, then Γ could be varied with e fixed. Variation of the action with respect to Γ would then give the Euler-Lagrange equations

$$D_\nu (e \, \tau_{\alpha\beta}{}^{\mu\nu}) = 0. \tag{3.5}$$

This would then be the first step in a straightforward derivation of the standard Einstein field equations. If Eq. (3.5) could be justified, then it would yield an explicit expression for Γ. This expression could then be used to eliminate Γ from the action. However, it is an immediate consequence of (2.14) and (2.16), and a special case of the important identity (2.31), that, when the matrix of vierbein coefficients is invertible, then

$$\frac{1}{2} \Gamma_\mu{}^{\alpha\beta} M_{\alpha\beta} = \partial_\mu , \tag{3.6}$$

which constrains the freedom of variation of Γ.

One way to deal with this is to abandon the geometrical interpretation (2.6) of the connection one-form as a section of the homomorphism from the local algebra to *diff(M)*, as well as the unification of vierbein and connection coefficients in the single complex ϕ. But the lack of internal coherence that results is perhaps the main cause of the dissatisfaction that is often expressed by saying that General Relativity is not a gauge theory. Indeed, the vierbein coefficients no longer seem to have a natural place in the theory, and this is why some authors favor a «purely geometrical» theory, by which is meant one in which no vierbein is introduced. A much more natural remedy is to enlarge the gauge algebra.

(ii) Take g to be the Poincaré algebra [16, 13]. Keep the tangent space $\widetilde{V}$ as above, with the same action (3.1) of the Lorentz subalgebra, and the translations acting trivially there. If (ℓ_α), $\alpha = 1, \ldots, 4$ is the usual basis for the translation subalgebra, then in the generic vector space V,

$$\ell_{\alpha\beta} \to S_{\alpha\beta}, \quad \ell_\alpha \to S_\alpha . \tag{3.7}$$

Since this representation is usually associated, in the field theoretical context, with the spin, it may seem natural to argue that the matrices S_α may be taken to vanish. This would be a mistake, however, since we are here concerned with structure, not representations. The operators Q_α, Eq. (2.10), take the form

$$Q_\alpha = e_\alpha + \frac{1}{2}\,\phi_q{}^{\beta\gamma}\, s_{\beta\gamma} + \psi_\alpha{}^\beta\, s_\beta, \tag{3.8}$$

$$e_\alpha = \frac{1}{2}\,\phi_\alpha{}^{\beta\gamma}\, M_{\beta\gamma} + \psi_\alpha{}^\beta\, M_\beta \tag{3.9}$$

The effect of having included the translations is very evident here. The coefficients $\phi_\alpha{}^{\beta\gamma}$ will play the same role as before, but the appearance of the new coefficients $\psi_\alpha{}^\gamma$ of the connection allows us to treat the coefficients of the Lorentz connection as independent field variables. We emphasize that the inverse vierbein coefficients are *not* identified with the coefficients of the connection one form Δ; instead of Eq. (3.6) we have

$$\frac{1}{2}\,\Gamma_\mu{}^{\alpha\beta}\, M_{\alpha\beta} + \Delta_\mu{}^\alpha\, M_\alpha = \partial_\mu. \tag{3.10}$$

with Γ and Δ defined, as in Eq. (2.16) by

$$\phi_\alpha{}^{\beta\gamma} = e_\alpha{}^\mu\, \Gamma_\mu{}^{\beta\gamma}, \quad \psi_\alpha{}^\beta = e_\alpha{}^\mu\, \Delta_\mu{}^\beta.$$

By (3.10), the translation connection Δ is determined by the Lorentz connection Γ, but there are no longer any constraints on the latter. Attempts to interpret the (inverse) vierbein directly as a connection oneform associated with the translations fail because vierbein coefficients do not transform like connection coefficients. The 10 coefficients $\phi_\alpha{}^A = (\phi_\alpha{}^{\beta\gamma}, \psi_\alpha{}^\beta)$ are independent of each other and form a Poincaré connection. The vierbein e, as well as the Poincaré connection oneform, are all determined by the $\phi_\alpha{}^A$s. Thus, without adding any additional variables, we avoid the problems that arise from having the vierbein double as a connection.

Torsion and curvature were introduced in Definition (2.22). The curvature (2.24) now has two parts:

$$R_{\alpha\beta}{}^A = \frac{1}{2}\, R(Lor)_{\alpha\beta}{}^{\gamma\delta}\, S_{\gamma\delta} + R\,(Tran)_{\alpha\beta}{}^\gamma\, S_\gamma.$$

The Lorentz curvature is still given by Eq. (3.2), and

$$R\,(Tran)_{\alpha\beta}{}^\gamma = \phi_\alpha{}^{\gamma\delta}\,\psi_{\beta\delta} + e_\alpha \psi_\beta{}^\gamma - (\alpha,\beta) - c_{\alpha\beta}{}^\delta\,\psi_\delta{}^\gamma. \tag{3.11}$$

The full curvature is of course a tensor field, taking values in a finite dimensional g-module. The action of g (the Poincaré algebra) in this module is nondecomposable, which finds expression in the fact that **the Lorentz curvature is a tensor while the $R(Tran)$ is not**. We can therefore (indeed we must) take the same action density as before, in terms of the contracted Lorentz curvature. Recall now the identity (2.30),

$$R_{\alpha\beta}{}^{A} M_A = 0 \tag{3.12}$$

which here says that $R(Tran)$ is determined by $R(Lor)$, so the question of whether the former plays any role in the theory is moot.

Having succeded in making Γ independent of e, we now return to the variational principle. (An easier way to study the content of the variational equations will be given later). Variation with respect to Γ gives Eq. (3.5), and this can be solved for Γ to yield

$$\Gamma_{\mu\nu}{}^{\lambda} = \gamma_{\mu\nu}{}^{\lambda} - f_\nu{}^{\alpha} \partial_\mu e_\alpha{}^{\lambda}, \quad f = e^{-1}, \tag{3.13}$$

$$\gamma_{\mu\nu\lambda} \equiv \frac{1}{2} (\partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}).$$

When Γ is eliminated from the action by means of this formula, then the result is the contracted curvature of the Cristoffel connection γ. This was elegantly demonstrated by Utiyama [15]. Eq. (3.13) can be re-arranged to read

$$D_\mu e_\alpha{}^{\nu} - \gamma_{\mu\lambda}{}^{\nu} e_\alpha{}^{\lambda} \equiv \nabla_\mu e_\alpha{}^{\nu} = 0. \tag{3.14}$$

The operator ∇ is the «total covariant derivative», where ∂_μ is replaced by parallell transport with respect to the metric (Cristoffel) connection. It extends to all tensor fields by means of Eq. (2.5) and the derivation rule.

The torsion tensor, Definition (2.22), vanishes when the Euler-Lagrange equations (3.5) hold. Notice a similarity between the expression (2.23) for the components of this tensor with the formula (3.11) for the translation part of the curvature; indeed, the latter reduces to the former when ψ is replaced by the unit matrix. This reminds us of a famous constraint, $R(Tran) = 0$ [17]. Of course, this restriction is not covariant, since as we have pointed out $R(Tran)$ is not a tensor. A related mistake that is often made is to confuse the connection one form Δ with the inverse vierbein: this sometimes arises from a desire to make do without introducing the vierbein in the first place, but only the complete Poincaré connection. This has always led to trouble, and the resolutions offered have never been satisfactory [18]. Usually, one is asked to accept a change in the interpretation, with attending change in transformation properties, in order to justify the choice of Lagrangian.

The vanishing of the torsion, together with the antisymmetry of $\phi_\alpha{}^{\beta\gamma}$ in the upper indices, gives us

$$2\phi_{\alpha\beta\gamma} = c_{\beta\gamma\alpha} - c_{\gamma\alpha\beta} - c_{\alpha\beta\gamma}. \tag{3.15}$$

This can be derived more directly from the expression

$$R_{\alpha\beta}{}^{\alpha\beta} = \phi_\alpha{}^{\alpha\gamma}\,\phi_{\beta\gamma}{}^{\beta} - \phi_\alpha{}^{\beta\gamma}\,\phi_{\beta\gamma}{}^{\alpha} + 2e_\alpha\phi_\beta{}^{\alpha\beta} - c_{\alpha\beta}{}^{\gamma}\,\phi_\gamma{}^{\alpha\beta}. \tag{3.16}$$

Variation of $\int eR$ with respect to $\phi_\alpha{}^{\beta\gamma}$ (with e fixed) gives immediately

$$c_{\beta\gamma}{}^{\alpha} + \phi_{\beta\gamma}{}^{\alpha} - \phi_{\gamma\beta}{}^{\alpha} = \delta_\beta{}^{\alpha}(\phi_{\delta\gamma}{}^{\delta} + e_\gamma{}^{\dagger}) - (\beta,\gamma),$$

$$e_\gamma{}^{\dagger} = e^{-1}\partial_\mu(e\,e_\gamma{}^{\mu})$$

Summing over $\alpha = \beta$ shows that the right hand side is zero and thus

$$c_{\beta\gamma}{}^{\alpha} + \phi_{\beta\gamma}{}^{\alpha} - \phi_{\gamma\beta}{}^{\alpha} = 0. \tag{3.17}$$

This derivation is independent of the factorization hypothesis (2.16).

Remark. For a fixed vierbein, the relation between torsion and Lorentz connection is one:one, so we can take the vierbein and the torsion as the independent variables. The curvature scalar is given by

$$R = A(t) - A(c),$$

$$A(t) = t_{\alpha\beta\gamma}\,t^{\alpha\beta\gamma} - 2\,t_{\alpha\beta\gamma}\,t^{\beta\gamma\alpha} - 4\,t_{\alpha\beta}{}^{\beta}\,t^{\alpha\gamma}{}_{\gamma}.$$

Now t is a tensor, so

$$A_{ab}(t) = t_{\alpha\beta\gamma}\,t^{\alpha\beta\gamma} + a\,t_{\alpha\beta\gamma}\,t^{\beta\gamma\alpha} + b\,t_{\alpha\beta}{}^{\beta}\,t^{\alpha\gamma}{}_{\gamma}$$

is a scalar for every a, b. An alternative action is therefore eR_{ab}, where

$$R_{ab} = A_{ab}(t) - A(c).$$

So long as the parameters a, b are such that A_{ab} is non-degenerate [A is degenerate if $a = -1$ or $+2$ or if $b = (a-2)/3$.], the variation of t will lead to $t = 0$ and the same reduced action. This remains true, for all practical purposes, even when fermions are present.

(iii) Take g to be the Poincaré algebra as before, but let $\widetilde{V}$, «tangent» space, be spinorial and endowed with a symplectic form η [19]. In $\widetilde{V}$ the translations act trivially, and $so(3,1)$ acts by real, four-dimensional symplectic matrices, a subalgebra of $sp(\eta)$. Instead of (3.1),

$$(\widetilde{S}_{ab})_c{}^{d} = \eta_{ac}\,\delta_b{}^{d} + (a,b), \quad \tilde{\phi}_{ab}{}^{c} = \phi_{ab}{}^{c}. \tag{3.18}$$

We use the letters a, b . . . for spinor indices and retain other conventions.

The components of torsion and curvature are

$$t_{abc} = c_{abc} + \phi_{abc} + \phi_{bac},$$

$$R_{ab}{}^{cd} = -\phi_{ae}{}^{c}\,\phi_{b}{}^{ed} + e_a\,\phi_b{}^{cd} + (ab) - c_{ab}{}^{e}\,\phi_e{}^{cd}.$$

The connection is fermionic and anticommuting, and so is the vierbein.

The invariant action is eR + possible torsion terms, and

$$R = -\phi_{ae}{}^{b}\,\phi_b{}^{ea} - \phi_{ae}{}^{a}\,\phi_b{}^{eb} + 2\,e_a\,\phi_b{}^{ab} - c_{ab}{}^{c}\,\phi_c{}^{ab}. \tag{3.19}$$

Further analysis, relegated to the Appendix, shows that one recovers, with the inclusion of suitable torsion terms, conventional metric gravity.

4. DE SITTER GRAVITY

Perhaps one does not expect a great deal of difference between gauging the De Sitter group or the Poincaré group. It is interesting, though, that here we do not have the option of gauging only the Lorentz subgroup. The De Sitter group is semisimple, and one corollary of that is that the curvature tensor is now fully reducible, in contrast with the Poincaré case.

We present two versions of gauged De Sitter gravity, since both are instructive. In both cases $g = so(3, 2)$; tangent space is either 4- or 5-dimensional. Since g must act in M, the manifold is the De Sitter hyperboloid in R^5, or a covering of it. If (y_a), $a = 0, 1, 2, 3, 5$ (latin indices have this range) are coordinates for R^5, then the hyperboloid is the locus

$$\delta^{ab}\,y_a\,y_b = \rho^{-1}, \quad (\delta^{ab}) = \mathrm{diag}\,(+ - - - +), \tag{4.1}$$

where ρ is the curvature constant, small but not zero. A basis for $so(3, 2)$ is given in terms of the action on M by (ℓ_{ab}), $a < b$,

$$\ell_{ab} \to M_{ab} = y_a\,\partial_b - y_b\,\partial_a. \tag{4.2}$$

We consider two possibilities for the tangent space.

(i) Take $\widetilde{V} = R^5$, with metric δ defined by (4.1), and g acting as $so(\delta)$ [20]. The operators of covariant differentiation are

$$Q_a = \frac{1}{2}\,\phi_a{}^{bc}\,L_{bc} = e_a + (1/2)\,\phi_a{}^{bc}\,S_{bc}, \tag{4.3}$$

$$e_a = \phi_a{}^{bc}\,y_b\,\partial_c = \phi_a{}^{c}\,\partial_c = e_a{}^{\mu}\,\partial_m,$$

where $\partial_c = \partial/\partial y^c$ and $\partial_\mu = \partial/\partial x^\mu$. (The embedding parameters y_a must not be confused with the coordinates x^μ, the choice of which will not be specified). The curvature coefficients take the form

$$R_{ab}{}^{cd} = \phi_a{}^{ce}\,\phi_{be}{}^{d} + e_a\,\phi_b{}^{cd} - (a, b) - c_{ab}{}^{e}\,\phi_e{}^{cd}. \tag{4.5}$$

The action of the local algebra g on ϕ and on R is reducible. This stems from the existence of an invariant function from M to $\widetilde{V}$, namely the function given by the imbedding map, with value $y_a(x)$. As far as ϕ is concerned, we restrict ourselves to the subspace defined by

$$y^a \phi_a{}^{bc} = 0.$$

A similar reduction using contractions against the upper indices is not $1g$-covariant, but nevertheless useful for computations. Define $\hat{\phi}$ by

$$\phi_c{}^{ab} = \hat{\phi}_c{}^{ab} + \rho(y^a \phi_c{}^b - y^b \phi_c{}^a); \tag{4.6}$$

then $\hat{\phi}$ is transverse to y on all three indices. The first term is associated with the local Lorentz algebra; that is, with the stabilizer of the point on the manifold indexed by y. It plays a role quite similar to the Lorentz connection in Poincaré gauge theory. The coefficients $\phi_a{}^b$ in the second term can be said to be associated with the local translations. The decomposition is not completely un-natural and we shall see that the «Lorentz connection» can be determined by the variational principle just as in the Poincaré case. Define an «inverse» to e:

$$e_a{}^\mu f_\nu{}^b = \delta_\nu{}^\mu$$

$$e_a{}^\mu f_\mu{}^b = \delta_a{}^b - \rho\, y_a y^b,$$

and eliminate $\hat{\phi}$ in favour of Γ, defined by

$$\hat{\phi}_c{}^{ab} = e_c{}^\mu \Gamma_\mu{}^{ab} = e_c{}^\mu f_\nu{}^a f_\lambda{}^b \Gamma_\mu{}^{\nu\lambda}. \tag{4.7}$$

Let R be the contracted curvature and take the action density to be R/e,

$$e = \epsilon^{abcde} \epsilon_{\mu\nu\lambda\rho} e_a{}^\mu e_b{}^\nu e_c{}^\lambda e_d{}^\rho y_e \rho^{1/2} \tag{4.8}$$

and introduce the metric

$$g_{\mu\nu} = \delta^{ab} e_a{}^\mu e_b{}^\nu. \tag{4.9}$$

Then a straightforward calculation shows that this is precisely the same action density as that of ordinary General Relativity.

This result is apparently not very well known, but in view of certain uniqueness theorems based on internal consistency it is not surprising. In pure De Sitter gravity as in pure Poincaré gravity, only *diff(M)* acts effectively. In the presence of spinor fields the two theories may be expected to differ, and they do. A cosmological term may be added in both theories, in neither case it is fixed automatically. What is eventually fixed by the inclusion of a cosmological term in the action is the choice of the background that can be used in perturbation theory. Comparison with the work of Mansouri and MacDowell [21], which we leave for the interested reader, will show that our inclusion of a vierbein avoids confusion, as well as the need to pass to the flat space limit at the end.

(ii) Let $g = so(3, 2)$ act in a four-dimensional tangent space $\widetilde{V}$ as $sp(\eta)$, with η a fixed nondegenerate antisymmetric matrix. In this section latin indices take the values 1, 2, 3, 4. We use a basis (ℓ_{ab}), $a \leqslant b$, defined by the action in $\widetilde{V}$, $\ell_{ab} \to \widetilde{S}_{ab}$,

$$(\widetilde{S}_{ab})_c{}^d = \eta_{ac}\,\delta_b{}^d + (a, b).$$

All the general formulas in Section 2, for the covariant derivatives Q_a, curvature and so on, apply (latin indices replacing early Greek). The plus sign must be taken in (2.22) and in the equations that follow since, by the normal spin-statistics association, the connection and the vierbein are fermionic. The component $R_{ab}{}^{cd}$ of the curvature are symmetric in both upper and lower components.

One can proceed as earlier to construct the contracted curvature and the action. The connection coefficients can be split into two sets, and those associated with the local Lorentz subalgebra (the stabilizer of x) can be eliminated by means of the variational equations, as before. The result appears to be essentially the same as in the Poincaré case. The most interesting question is whether the fermionic vierbein field is subject to a direct physical interpretation.

If the goal is a gauge theoretic framework from which to approach quantum field theory on De Sitter space, then the cosmological term must be included. Since the final theory is to be De Sitter covariant, it is natural to gauge this algebra. One does not have the option of gauging only a subalgebra, since no subalgebra distinguishes itself in a natural way. In the case of Minkowski space, the Poincaré algebra does have a such a natural subalgebra, but to make use of an option that is available in this singular limit only, seems to us highly unnatural. In other words, when placed in a larger context, the question of whether Poincaré gravity should be seen as a gauge theory of the Poincaré algebra or as a gauge theory of the Lorentz algebra has only one reasonable answer. Thus, we should not have been surprised to find, as we did, that the choice of the Poincaré algebra is the only workable option. In any case, it is the only choice consistent with stability of physical theory.

5. POINCARE SUPERGRAVITY

Here we hope to show that all this attention to structure has been useful. Instead of a manifold and a Lie algebra, we are now dealing with a super manifold and a super Lie algebra, but this is not the main reason why supergravity is a difficult theory to understand as a gauge theory. Instead, the trouble arises m the structure of the superalgebra g. The super manifold M is defined in ms of R^4 and the Grassmann algebra with 4 generators [22]. For the Lie ıer algebra g we use a basis $(\ell_{\alpha\beta})$, $\alpha < \beta$ (with $\ell_{\alpha\beta} = -\ell_{\beta\alpha}$), for the Lorentz ›algebra, (ℓ'_α) for the translations and (ℓ_a) for the super translations, $a, b, \ldots$ ng four-spinor indices and $\alpha, \beta, \ldots$ being four-vector indices. The Lorentz ebra acts in spinor spaces by matrices $\Sigma_{\alpha\beta}$, and the structure is given by

$$[\ell_a, \ell_b] = (\gamma^5\gamma^\alpha)_{ab}\, \ell'_\alpha,$$

$$[\ell_a, \ell'_\beta] = 0, \quad [\ell'_\alpha, \ell'_\beta] = 0,$$

$$[\ell_{\alpha\beta}, \ell'_\gamma] = \delta_{\beta\gamma}\, \ell'_\alpha - \delta_{\alpha\gamma}\, \ell'_\beta, \quad [\ell_{\alpha\beta}, \ell_a] = (\Sigma_{\alpha\beta})_a{}^b\, \ell_b,$$

ere $(\gamma^5\gamma^\alpha)_{ab} = (\gamma^5\gamma^\alpha)_a{}^c\, \eta_{cb}$ is symmetric, η the symplectic metric. We notice ıt this action of g on itself is nondecomposable, with two uncomplemented 'ariant subspaces; one spanned by all the translations and the other spanned the ordinary translations in R^4.

As in Section 2, g acts by vector fields M_A in M, by matrices S_A in the generic nodule V, and by the operators $L_A = M_A + S_A$ in $\mathscr{F}(M, V)$. The local super- ;ebra $\mathrm{l}g$ acts by operators $\Lambda^A L_A$. In the case when V is the trivial g-module ıs action reduces to that of *diff*(M), but the local gauge algebra is much larger an *diff*(M).

The tangent space $\widetilde{V}$ has to be a g-module; we suppose that it is finite dimen- ›nal. All nontrivial, finite dimensional representations are either nonfaithful nondecomposable, or both. The simplest ones are the four-dimensional spinor d vector representations of the Lorentz algebra, with all the translations acting vially. However, the vielbein matrix is in this case rectangular and therefore ıt invertible, which causes problems. Perhaps the simplest way to overcome ch difficulties is to begin with a square vielbein matrix and look for a reduction ›pearing subsequently. We therefore suppose that tangent space is 8-dimen- ›nal, more precisely a graded vector space of dimension 4 + 4, the action the Lorentz subalgebra being the sum of the two 4-dimensional representa- ›ns [23]. The superalgebra can act on this space in several different ways, :rhaps the most natural is that obtained by analogy with the adjoint action ı the invariant subalgebra that consists of all the translations. If (ϕ_a, ϕ'_α) are e components of $\phi \in V$, then this gives, in particular:

$$(\widetilde{S}_a\phi)_b = 0, \quad (\widetilde{S}_a\phi)'_\alpha = k(\gamma_5\gamma_\alpha)_a{}^b\, \phi_b. \qquad \text{.1)}$$

A parameter k has been included; if it is taken to vanish then the module :comes completely reducible. Another possibility is the dual,

$$(\widetilde{S}\ \phi)_\cdot = k(\gamma^5\gamma^\alpha)\ \cdot\ \phi'\ ,\ (\widetilde{S}\ \phi)'\ = 0. \qquad \text{; 2)}$$

If (5.1) is adopted, then the condition $\phi_a = 0$ defines a submodule, while $\phi'_\alpha = 0$ defines a submodule only if $k = 0$. (Conversely if (5.2) holds).

With an 8-by-8 vielbein matrix one encounters no obstacles in copying the procedure that was followed in the treatment of ordinary gravity, in Section 3, up to the point of choosing an action functional. The curvature tensor takes values in $(V \wedge V) \otimes V_g$ and breaks into nine parts. With respect to V_g we have a Lorentz curvature and two kinds of translation curvature, but only the Lorentz part is a sub-tensor (no constraint on other components is consistent with invariance), and only the Lorentz part can be contracted to form a scalar. With respect to $V \wedge V$ the only sub tensor is the bi-spinor part if (5.1) holds and the bi-vector part if (5.2) is adopted. The only (linear) scalar function is the Lorentz curvature constracted as follows:

(5.3) $$R = \frac{1}{2} R_{ab}{}^{\alpha\beta} (\Sigma_{\alpha\beta})^{ab}, \qquad \text{if (5.1) holds.}$$

(5.4) $$R' = R'_{\alpha\beta}{}^{\alpha\beta}, \qquad \text{if (5.2) holds.}$$

The action density is assumed to be of the form eR, with the density factor e constructed out of the vielbein. Let us postpone the question of the density factor.

We want first to vary the action with respect to ϕ_a and ϕ'_α. A formula for R is obtained from Eq.s (3.19) and (5.3),

(5.5) $$R = (\Sigma_{\alpha\beta})^{ab} \left[\phi_a{}^{\alpha\gamma} \phi_{b\gamma}{}^{\beta} + e_a \phi_b{}^{\alpha\beta} - \frac{1}{2} c_{ab}{}^{e} \phi_e{}^{\alpha\beta} - \frac{1}{2} c_{ab}{}^{\gamma} \phi'_\gamma{}^{\alpha\beta} \right]$$

or from Eq.s (3.2) and (5.4),

(5.5') $$R' = \phi'_\alpha{}^{\alpha\gamma} \phi'_{\beta\gamma}{}^{\beta} - \phi'_\alpha{}^{\beta\gamma} \phi'_{\beta\gamma}{}^{\alpha} + 2e'_\alpha \phi'_\beta{}^{\alpha\beta} - c_{\alpha\beta}{}^{\gamma} \phi'_\gamma{}^{\alpha\beta} - c_{\alpha\beta}{}^{c} \phi_c{}^{\alpha\beta}.$$

Let us now fix our attention on the first case.

The expression (5.5) is dominated by the spinorial part of the connection, to the almost total exclusion of the vector part. The last term in (5.5) is nevertheless required for invariance, nor is it possible to eliminate ϕ' from the action by adding torsion terms. Variation of the vectorial connection gives the covariant constraint

(5.6) $$(\Sigma_{\alpha\beta})^{ab} c_{ab}{}^{\gamma} = 0.$$

This is equivalent [since $c_{ab}{}^{\gamma} = t_{ab}{}^{\gamma}$, see below] to

(5.7) $$(\Sigma_{\alpha\beta})^{ab} t_{ab}{}^{\gamma} = 0, \text{ or } t_{ab}{}^{\gamma} = (\gamma^5 \gamma^\beta)_{ab} K_\beta{}^{\gamma}.$$

with arbitrary coefficients $K_\beta{}^\alpha$. Eq. (5.6) reduces, in the simplest case when $K_\beta{}^\alpha = \delta_\epsilon{}^\alpha$, to the famous torsion constraint introduced into superspace supergravity by Wess and Zumino [5]. It has not, until now and as far as we know, been related to an interesting variational principle. To see that (5.6) is equivalent to (5.7), we note that the components of torsion are, in the case considered; that is, when (5.1) holds:

(5.8)
$$t_{ab}{}^\gamma = c_{ab}{}^\gamma, \qquad t_{ab}{}^c = c_{ab}{}^c + \tilde{\phi}_{ab}{}^c + (a, b)$$
$$t_{a\beta}{}^\gamma = c_{a\beta}{}^\gamma - \phi_{a\beta}{}^\gamma, \qquad t_{a\beta}{}^c = c_{a\beta}{}^c + k\phi_a{}^d(\gamma_5\gamma_\beta)_d{}^c$$
$$t_{\alpha\beta}{}^\gamma = c_{\alpha\beta}{}^\gamma + \phi'_{\alpha\beta}{}^\gamma - (\alpha, \beta), \quad t_{\alpha\beta}{}^c = c_{\alpha\beta}{}^c + [k\phi'_\alpha{}^d(\gamma_5\gamma_\beta)_d{}^c - (\alpha, \beta)]$$

These expressions have been so arranged that the action of the super translations leaks downwards and rightwards. (The dimension increases in the same sense). The only sub-tensor among them (except for the supertrace, see below) is the first one; this is also the only one that is completely determined by the holonomy coefficients. The constraint (5.7) is thus covariant.

Variation of the spinoral part of the connection now leads to equations that are covariant provided that the constraints that were obtained by variation of the vectorial part are satisfied. This inter-relationship between constraints and covariance is a direct consequence of the nondecomposable action of the superalgebra in tangent space.

None of the Euler-Lagrange equations involve the vectorial connection, which thus remains completely arbitrary. This would be unsatisfactory if we had a theory in which these coefficients play a role, but actually they do not appear anywhere except in the present context. All supersymmetric theories in superspace have been formulated entirely in terms of spinorial covariant derivatives, a curious fact that fits in very well with the present view of supergravity. Some components of the spinorial connection also remain undetermined, just as happened in the case of the spinorial formulation of ordinary gravity. The part of (5.5) that contains ϕ has exactly the same form as the curvature scalar (3.19), as is seen when the components of the connection are transformed from the spinorial basis (3.19) to the vectorial basis used here for the Lorentz algebra. Further remarks are relegated to the Appendix.

Turning briefly to the other possibility, Eq. (5.5'), we at once recognize the expression for the ordinary Poincaré contracted curvature. The Euler-Lagrange equations obtained by variation of ϕ and ϕ' are,

$$t_{\alpha\beta}{}^\gamma = 0, \text{ and } c_{\alpha\beta}{}^c = 0.$$

To verify that this is covariant we again write down the expression for the torsion, assuming (5.2):

$$t_{\alpha\beta}{}^{c} = c_{\alpha\beta}{}^{c}, \qquad t_{\alpha\beta}{}^{\gamma} = c_{\alpha\beta}{}^{\gamma} + (\phi'_{\alpha\beta}{}^{\gamma} - \alpha, \beta)$$

$$t_{a\beta}{}^{c} = c_{a\beta}{}^{c} - \phi'_{\beta a}{}^{c}, \qquad t_{a\beta}{}^{\gamma} = c_{a\beta}{}^{\gamma} - \phi_{a\beta}{}^{\gamma} \quad k\phi'_{\beta}{}^{d}(\gamma^{5}\gamma^{\gamma})_{da},$$

$$t_{ab}{}^{c} = c_{ab}{}^{c} + (\phi_{ab}{}^{c} + a, b), \quad t_{ab}{}^{\gamma} = c_{ab}{}^{\gamma} + [k\phi_{a}{}^{d}(\gamma^{5}\gamma^{\gamma})_{db} + (a, b)] .$$

The arrangement is according to the same principles as in (5.8). (The dimension now decreases). This again confirms that the constraints obtained are covariant, as of course they must be.

We have thus seen that there may be two quite different formulations of super gravity. The first one is very likely to be closely related to the superspace theory of Wess and Zumino [5], and to contain the component form of supergravity. Perhaps the simplest way to confirm this is to investigate the linear approximation; we shall do this in the next sections.

6. THE LINEAR APPROXIMATION

Supergravity was discovered quite some time ago. If we take interest in a new derivation of it, it is for the purpose of testing the generality and the productivity of a proceedure («philosophy») that claims to be rigid enough to be used without ambiguity in the search for new theories. The formulation of supergravity as a superspace gauge theory was accomplished already [5, 6, 7], by a tour de force of improvisation and inventiveness. Here, no improvisation has been required (so far), and therein lies the alleged advantage of our approach. It is thus crucial to make sure that the «general structure» of Section 2 really incompasses superspace supergravity, without compromise of integrity in the application. The simplest way to do this is to examine the linear approximation, in which the full kinematical structure must stand revealed.

We consider only the Wess-Zumino version. The first step is to find an exact solution around which to expand. The simplest possibility is a solution for which the Lorentz connection vanishes; then the field equations reduce to the torsion constraints. We require a solution that is stable with respect to global supersymmetry. Invariance under Lorentz transformations will be manifest, and the translations will act trivially, so it remains only to test the super translations. The variations of the non-vanishing part of the connection are given by Eq. (2.13) (the second, «affine» term is zero for global transformations), and by Eq. (5.1); they must all vanish:

$$(\delta_a\phi)_b{}^{c} = M_a\phi_b{}^{c} = 0, \quad (\delta_a\phi)_b{}^{\gamma} = M_a\phi_b{}^{\gamma} + (\gamma^{5}\gamma^{\gamma})_{ac}\phi_b{}^{c} = 0,$$

$$(\delta_a\phi)'_{\beta}{}^{c} = M_a\phi_{\beta}{}^{c} + k(\gamma_5\gamma_{\beta})_a{}^{b}\,\phi_b{}^{c} = 0, \qquad (\delta_a\phi)'_{\beta}{}^{\gamma} = M_a\phi'_{\beta}{}^{\gamma} - (\gamma^{5}\gamma^{\gamma})_{ac}\phi'_{\beta}{}^{c} + k(\gamma_5\gamma_{\beta})_a{}^{b}\phi_b{}^{\gamma} = 0.$$

Notice once again the leaking towards right and down. We must solve in the same order; the simplest solution is

(6.1) $$\phi_b{}^c = \delta_b{}^c, \qquad \phi_b{}^\gamma = -\theta^\gamma{}_b,$$
$$\phi'_\beta{}^c = -k\theta_\beta{}^c, \quad \phi'_\beta{}^\gamma = (1 + k\theta^2)\delta_\beta{}^\gamma.$$

Here we used the abbreviation

$$\theta_{\alpha b} = (\theta\gamma_5\gamma_\alpha)_b$$

and the conventional expressions for the vector fields that define the action of g:

(6.2) $$M_a = \partial_a + \frac{1}{2}\theta^\mu{}_a\partial_\mu, \quad M'_\alpha = \partial_\alpha.$$

Thus

(6.3) $$e_b = M_b - \theta^\gamma{}_b M'_\gamma = \partial_b - \frac{1}{2}\theta^\mu{}_b\partial_\mu,$$
$$e'_\beta = (1 + k\theta^2)M'_\beta - k\theta_\beta{}^c M_c,$$

which includes the familiar flat space spinorial «covariant derivative».

To invert these relations we have the identity

(6.4) $$\theta_\alpha{}^c\,\theta^\beta{}_c = \delta_\alpha{}^\beta\,\theta^2, \quad \theta^2 = \theta^c\theta_c.$$

The result is

(6.5) $$M'_\beta = e'_\beta + k\theta_\beta{}^c\,e_c,$$
$$M_b = \theta^\alpha{}_b\,e'_\alpha + (\delta_b{}^c + k\theta^\alpha{}_b\theta_\alpha{}^c)\,e_c.$$

To verify the torsion constraints we calculate

$$[e_a, e_b] = -(\gamma^5\gamma^\gamma)_{ab}M'_\gamma = c_{ab}{}^\gamma e'_\gamma + c_{ab}{}^c e_c,$$

which in view of (6.5) means that

(6.6) $$c_{ab}{}^\gamma = -(\gamma^5\gamma^\gamma)_{ab},$$
$$c_{ab}{}^c = -k(\gamma^5\gamma^\gamma)_{ab}\,\theta_\gamma{}^c.$$

The first formula is in accord with the constraint (5.7), and even with the particular choice of Wess and Zumino [5]. The second expression satisfies the constraint (3.24). Both torsion constraints are thus satisfied, and therewith all the variational equations. Eq. (5.10) reduces to $\partial_\mu e = 0$, and the super determinant of the achtbein matrix is equal to unity.

The linear approximation is a perturbation of (6.3), we express it by

$$e_b = q_b + E_b{}^c\, \partial_c + E_b{}^\gamma\, \partial'_\gamma . \tag{6.7}$$

Here, and from now on, q_b and q'_β stand for the fixed point vector fields (6.3) the former being the flat space spinorial «covariant derivative». This leads in the linear approximation to

$$[e_a, e_b] = -(\gamma^5 \gamma^\gamma)_{ab} M'_\gamma + F_{ab}{}^c\, \partial_c + F_{ab}{}^\gamma\, \partial'_\gamma , \tag{6.8}$$

with

$$F_{ab} = q_a E_b + (a, b). \tag{6.9}$$

The contribution of the first term satisfies the constraint (5.7). If we denote by bold face the linear part of the coefficients, then

$$\mathbf{c}_{ab}{}^c = F_{ab}{}^c + k\mathbf{c}_{ab}{}^\gamma\, \theta_\gamma{}^c ,$$

$$\mathbf{c}_{ab}{}^\gamma = \frac{1}{2} F_{ab}{}^d\, \theta^\gamma{}_d + F_{ab}{}^\gamma .$$

The constraints (5.7) becomes

$$\mathbf{c}_{ab}{}^\gamma = (\gamma^5 \gamma^\alpha)\, K_\alpha{}^\gamma , \tag{6.10}$$

with K arbitrary except for the requirement that Eq. (6.9) be integrable. The other torsion constraint, Eq. (5.11), serves mainly to constrain the Lorentz connection, except for the implication (3.24) on the holonomy coefficients. In view of (6.10), this becomes

$$(\psi^\alpha \gamma^\beta)_a{}^b\, (\widetilde{S}_{\gamma\delta})^{cd}\, F_{cd}{}^a = 0,$$

with ψ satisfying $\gamma_\alpha \psi^\alpha = 0$ but otherwise arbitrary.

We should now substitute this back into the action, eliminating the connection in favour of the vielbein, and finally vary the action with respect to the latter, remembering the constraints. We have not yet done this. Instead we have made a preliminary investigation of the coupling to matter that we feel gives strong support to the conjecture that the local gauge theory constructed here is in fact related to super gravity. This is reported in Section 7.

7. MATTER LAGRANGIAN

Let Φ be a scalar superfield,

$$\Phi = \phi + \theta\psi + \theta^2 A + (1/2)\, \theta^{ab} B_{ab} + \theta^2 \theta \chi + \theta^4 F,$$

with B traceless. The invariant Lagrangian is

(7.1) $$L_m = \int dx\, d\theta\, e\left(\frac{1}{2}\,\eta^{ba}(e_a\Phi)(e_b\Phi) + m\Phi\Phi\right).$$

In the zero'th approximation, when $e = q$, this reduces to

(7.2) $$L_{mo} = \int dx\left(-\frac{1}{8}\,(\psi\partial\!\!\!/\chi + \chi\partial\!\!\!/\psi) - \frac{1}{32}\,\psi_{,\mu}\psi_{,\mu} - \frac{1}{2}\,\chi\chi - \frac{m}{2}\,\psi\chi\right.$$
$$+ 8AF + \frac{1}{4}\,\phi_{,\mu}A_{,\mu} + \frac{1}{16}\,tr(B\partial\!\!\!/B)$$
$$\left.+\, m\left[2\phi F + A^2 + \frac{1}{16}\,tr\,BB\right]\right).$$

The spinorial field equations

(7.3) $$\delta\psi\left[-\frac{1}{4}\,\partial\!\!\!/\chi + \frac{1}{16}\,\partial^2\psi - \frac{m}{2}\,\chi\right] = 0,$$

(7.4) $$\delta\chi\left[-\frac{1}{4}\,\partial\!\!\!/\psi - \chi - \frac{m}{2}\,\psi\right] = 0,$$

can be rearranged to give

(7.5) $$4\chi = -m\psi,\ (\partial\!\!\!/ + m)\psi = 0.$$

Variation of the scalar fields,

(7.6) $$\delta\phi\left[-\frac{1}{4}\,\partial^2 A + 2mF\right] = 0,$$
$$\delta A\left[8F - \frac{1}{4}\,\partial^2\phi + 2mA\right] = 0,$$
$$\delta F[8A + 2m\phi] = 0$$

reduces to

(7.7) $$8F = -mA,\ 4A = -m\phi,\ (\partial^2 + m^2)\phi = 0.$$

Finally,

(7.8) $$\delta B^{ab}\left[\frac{1}{8}\,(\partial\!\!\!/ - m)B\right]_{ab} = 0$$

can be expressed in terms of a vector field and a pseudo-scalar field:

$$B^\nu = tr(\gamma^\nu B), \quad 4B = -\gamma^\nu B_\nu,$$

where, exceptionally, ν takes 5 values, with the result that

$$(7.9) \qquad mB_\mu = \partial_\mu B_5, \quad mB_5 = -\partial_\mu B_\mu.$$

Thus one finds that the wave equations, though they at first appear to be of the dipole type, actually describe nothing more than the irreducible, massive super-multiplet. This is a minor miracle, for higher order wave equations have a marked tendency to describe unwanted degrees of freedom.

This approach to the scalar superfield appears to be very simple and straightforward [24]. It is not the usual one, and for valid reasons. It is not at all evident that interactions can be introduced without spoiling the miraculous absence of ghosts, and it becomes necessary as well as interesting to understand what mechanism might be at work to bring it about. In the case of the minimal interaction with an external (super–) gravitational field, defined by the Lagrangian (7.1), we suspect that the absence of ghosts may be guaranteed by the torsion constraints. This is not absurd, since these constraints are actually field equations, derived by variation of the Lorentz connection. The Lorentz connection does not appear in the matter lagrangian (7.1); remember that the connection was varied with the vielbein fixed, so L_m is not affected and the torsion constraints are the same as in the absence of matter. The idea that torsion constraints ensure self consistency in the matter sector is not new [25]. Recall also that constraints on the background super-gravity field are connected to the absence of ghosts in superstrings and super membranes [26].

To verify that the torsion constraints do in fact play this role would seem to require very extensive calculations (or better insight). We prefer to merely nibble at the problem at first, hoping that the indications will encourage a greater effort. We shall limit the scope as follows. The supergravity background will be treated in the linear approximation,

$$e_a = q_a + E_a,$$

to first order in the perturbation E_a, and this vector field will be taken to be of a very special form.

We ignore all of E_a, except the term

$$(7.10) \qquad E_a = \epsilon_{ab}{}^{\mu}\, \theta^b\, \partial_\mu,$$

with the coefficients ϵ constant. This is in fact the term that couples to the highest derivatives of the matter fields, and the first one that needs to be controlled. The first order correction to L_{mo} is then

(7.11) $$L_{m\,1} = \int dx\, d\theta\, ((q^a \Phi)(E_a \Phi) + \delta(e)\, \mathscr{L}_{mo}),$$

where $\delta(e)$ is the first order correction to the density factor. We find after integration over θ that the first term contributes the following spinorial interactions:

(7.12) $$\int dx \left(\frac{1}{16} (\gamma^5 \gamma^\mu)^{ab} \, [-2\epsilon_{ac}{}^\nu \psi^c{}_{,\mu} \psi_{b,\nu} + \epsilon_{ab}{}^\nu \psi^c{}_{,\mu} \psi_{c,\nu}] - \frac{1}{4} \epsilon_{ab}{}^\mu [2\chi^b \psi^a{}_{,\mu} - \eta^{ab} \chi^c \psi_{c,\mu}] \right).$$

The field equations (7.3) and (7.4) are thereby modified to

(7.13) $$(-4\partial\!\!\!/\chi + \partial^2 \psi - 8m\chi)_c = -(\gamma^5\gamma^\mu)^{ab} (2\epsilon_{ac}{}^\nu \psi_{b,\mu\nu} + 2\eta_{bc}\, \epsilon_{ad}{}^\nu \psi^d{}_{,\mu\nu} - 2\epsilon_{ab}{}^\nu \psi_{c,\mu\nu}) + 8\epsilon_{ca}{}^\mu \chi^a{}_{,\mu} + 4\eta^{ab}\, \epsilon_{ab}{}^\mu \chi_{c,\mu},$$

(7.14) $$(-\partial\!\!\!/\psi - 4\chi - 2m\psi)_a = 2\epsilon_{ba}{}^\mu \psi^b{}_{,\mu} - \eta^{bc}\, \epsilon_{bc}{}^\mu \psi_{a,\mu}.$$

The problem is to determine, under what conditions on ϵ, these equations describe the same number of degrees of freedom as they do when $\epsilon = 0$. This is the case if they allow us to express, to first order in ϵ, the field χ in terms of ψ and its first derivatives.

When $\epsilon = 0$, the constraint is found by applying the operator $-\partial + 2m$ to the second equation and adding it to the first equation. To first order in ϵ this gives

(7.15) $$-4m(4\chi + m\psi)_c = -2\epsilon_{ac}{}^\nu (\partial\!\!\!/\psi)^a{}_{,\nu} - 2(\gamma^5\gamma^\mu)_c{}^a (\epsilon_{ad}{}^\nu + \epsilon_{da}{}^\nu)\, \psi^d{}_{,\mu\nu} + 2(\gamma^5\gamma^\mu)^{ab}\, \epsilon_{ab}{}^\nu \psi_{c,\mu\nu} + \eta^{bc}\, \epsilon_{bc}{}^\mu (\partial\!\!\!/\psi)_{c,\mu} + 8\epsilon_{ca}{}^\mu \chi^a{}_{,\mu} + 4\eta^{ab}\, \epsilon_{ab}{}^\mu \chi_{c,\mu} + 2m[2\epsilon_{ac}{}^\mu \psi^a{}_{,\mu} - \eta^{ab}\, \epsilon_{ab}{}^\mu \psi_{c,\mu}].$$

On the right hand side we can use the zero'th order field equations to simplify, and especially to reduce the number of derivatives, obtaining

$$= -2(\gamma^5\gamma^\mu)_c{}^a (\epsilon_{ad}{}^\nu + \epsilon_{da}{}^\nu)\, \psi^d{}_{,\mu\nu} + 2(\gamma^5\gamma^\mu)^{ab}\, \epsilon_{ab}{}^\nu \psi_{c,\mu\nu} + 8(\epsilon_{ac}{}^\nu + \epsilon_{ca}{}^\nu)\, \chi^a{}_{,\nu} + 4m\, [2\epsilon_{ac}{}^\mu \psi^a{}_{,\mu} - \eta^{ab}\, \epsilon_{ab}{}^\mu \psi_{c,\mu}].$$

Except in the mass term, only the symmetric part of ϵ^μ appears. This is precisely the part that appears in the torsion constraint. According to Eqs. (6.9-10), the term without θ's, we must have

$$\epsilon_{ab}{}^\nu + \epsilon_{ba}{}^\nu = K^{\nu\lambda} (\gamma_5 \gamma_\lambda)_{ab}.$$

Inserting this into (7.16) we get

$$= -2K^{\nu\lambda}(\gamma_5\gamma_\lambda)_c{}^d(\partial\!\!\!/\psi + 4\chi)_{d,\nu} + 4m\,[2\epsilon_{ac}{}^\mu\,\psi^a{}_{,\mu} \quad \eta^{ab}\,\epsilon_{ab}{}^\mu\,\psi_{c,\mu}]$$

$$= 4m\,[K^{\nu\lambda}(\gamma_5\gamma_\lambda)_c{}^d\,\psi_{d,\nu} + 2\epsilon_{ac}{}^\mu\,\psi^a{}_{,\mu} \quad \eta^{ab}\,\epsilon_{ab}{}^\mu\,\psi_{c,\mu}]$$

and thus, finally

$$4\chi_c + m\psi_c + (\epsilon_{ac}{}^\mu - \epsilon_{ca}{}^\mu)\,\psi^a{}_{,\mu} - \eta^{ab}\,\epsilon_{ab}{}^\mu\,\psi_{c,\mu} = 0. \tag{7.16}$$

The second order derivatives have cancelled, as a result of the torsion constraint. Eq. (7.16) reduces the number of degrees of freedom, just as effectively as in the case when $\epsilon = 0$. The effectiveness of the torsion constraint (5.7) in helping to avoid ghosts seems to be established.

Eliminating χ between (7.14) and (7.16) one gets the first order correction to the Dirac equation for ψ,

$$[\partial\!\!\!/_c{}^d + K^{\nu\lambda}(\gamma_5\gamma_\lambda)_c{}^d\,\partial_\nu]\,\psi_d + m\,\psi_c = 0.$$

Thus, the Dirac γ-matrices have been modified,

$$\gamma^\nu \to \gamma^\nu + K^{\nu\lambda}\,\gamma_\lambda = \hat{\gamma}^\nu.$$

The anticommutator is

$$[\hat{\gamma}^\nu, \hat{\gamma}^\mu]_+ = -2(\delta^{\nu\mu} + K^{\mu\nu} + K^{\nu\mu}),$$

which tells us that the symmetric part of K is related to the gravitational metric. A parallel analysis of the bosonic sector confirms these conclusions.

8. THE DIMENSION OF TANGENT SPACE

The choice of a tangent space of dimension 4 + 4 for super-gravity is not so natural as it may appear at first sight. First of all, only spinorial covariant derivatives seem to be important. In addition, this space does not have a natural extension to De Sitter super-gravity. By far, the most attractive possibility for that theory is a 4 + 1 dimensional osp(1/4) module, in terms of which matter described by scalar superfields, as well as super electrodynamics, find a very natural setting [25]. This fact is something that cannot be ignored, for it would be extremely unexpected to find a good theory in flat space, not deformable by the introduction of a cosmological constant. (Some people do hope for just that to happen!). It behooves us, therefore, to try to eliminate the vectorial vielbein, the role of which in super-gravity is very marginal anyhow, and try to formulate the theory entirely with just a 4-spinor tangent space. In fact, this is not difficult.

The problem that arises, whenever the dimension of $\widetilde{\Gamma}$ is deficient relative to the dimension of the manifold, is how to define torsion and curvature. In general,

$$[Q_a, Q_b] = [e_a, e_b] + e_a \phi_b + $$
$$+ e_b \phi_a + [\phi_a, \phi_b] + (\tilde{\phi}_{ab}{}^c + a, c)\, Q_c .$$

If the first term can be expressed as $c_{ab}{}^c\, e_c$, then this can be rearranged to read

$$(c_{ab}{}^c + \tilde{\phi}_{ab}{}^c + \tilde{\phi}_{ba}{}^c)\, Q_c + R_{ab} = t_{ab}{}^c\, Q_c + R_{ab} .$$

If not, and that is the case of interest, then we parameterize the ambiguity by «borrowing» additional connections ϕ'_α, from which we get additional vector fields $e'_\alpha = \phi_\alpha{}^A M_A$. We assume that the full set (e_a, e'_α) is a basis for TM_x at each point x of M, which means that there is a unique expansion

$$[e_a, e_b] = c_{ab}{}^c\, e_c + c_{ab}{}^\gamma\, e'_\gamma .$$

This allows us to define, for each choice of the ϕ'_s, a torsion, a curvature, and an action. Of course, the action is not independent of these borrowed fields, but the ambiguity is neatly avoided by declaring that the action must be stationary with respect ot variations of them. That is, in fact, exactly what we did. The only thing that was not clear was that the nature of the $\phi'_\alpha s$ is totally irrelevant, so long as there are enough of them. They play the role of Lagrange multipliers, and the resulting constraints remove the ambiguity inherent in the definition of the action.

Let us look at all this in terms of a different parameterization. We factorize:

$$Q_a = e_a{}^\mu\, (\partial_\mu + \Gamma_\mu).$$

(Indices μ, ν, λ, ρ refer to TM). When $dim(\tilde{V})$ is less than $dim(M)$, as we suppose, then this is ambiguous; the $\Gamma's$ are defined only modulo variations $\delta\Gamma$ satisfying

$$(8.1) \qquad e_a{}^\mu\, \delta\Gamma_\mu = 0.$$

The action

$$(8.2) \qquad \int dx\, d\theta\, e_a{}^\mu\, e_b{}^\nu\, ([\Gamma_\mu, \Gamma_\nu] + (d\Gamma)_{\mu\nu})^{\alpha\beta}\, (\Sigma_{\alpha\beta})^{ab},$$

is required to be stationary with respect to all variations. In particular, if $\delta\Gamma$ is of the type that satisfies (8.1), then this gives the constraint

$$(8.3) \qquad \int dx\, d\theta\, (\delta\Gamma_\mu)^{\alpha\beta}\, [e_a, e_b]^\mu\, (\Sigma_{\alpha\beta})^{ab} = 0,$$

which is just the constraint (5.7). We conclude that dimensionally deficient «tangent» spaces are not only possible but may actually be of special interest. In particular, they may be a nice way to understand all the torsion type constraints on super gravities and super Yang-Mills gauge theories.

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APPENDIX

Here we first complete the analysis of ordinary gravity in the formulation based on a spinorial tangent space, then investigate analogous problems in supergravity.

Since the Lorentz algebra is only a subalgebra of $sp(\eta)$ it is convenient to use the convention of dotted and undotted indices, as we shall do from now on. The components of torsion are given by

$$t_{abc} = c_{abc} + \phi_{abc} + \phi_{bac},$$

$$t_{\dot a bc} = c_{\dot a bc} + \phi_{\dot a bc},$$

$$t_{ab\dot c} = c_{ab\dot c},$$

and three other formulas obtained from these by«conjugation»: exchanging dotted for undotted indices. Note that $\phi_{ab\dot c} = 0$, since the algebra is $so(3, 1)$ and not $sp(\eta)$, and that the following components of torsion are determined by the vierbein

(A.1) $$c_{ab\dot c} = t_{ab\dot c}, \quad c_{\dot a bc} - c_{\dot a cb} = t_{\dot a bc} - t_{\dot a cb}.$$

The non-zero components of the connection are

$$2\phi_{abc} = (t - c)_{abc} - (t - c)_{bca} + (t - c)_{cab},$$

$$\phi_{\dot a bc} = (t - c)_{\dot a bc},$$

and conjugates. In this notation the expression for the curvature scalar becomes

(A.2) $$R = -\phi_{ac}^{\ \ b}\,\phi_{\dot b}^{\ ac} - \phi_{ac}^{\ \ a}\,\phi_{b}^{\ cb} + 2\,e_a\,\phi_b^{\ ab} +$$
$$c_{ab}^{\ \ c}\,\phi_c^{\ ab} - c_{ab}^{\ \ \dot c}\,\phi_{\dot c}^{\ ab} + \textit{conj. terms.}$$

In terms of t and c, with $t_a{}' = t_{ab}{}^{b} + t_{ab}{}^{\dot b}$,

$$R = -\frac{1}{2}(t - c)_{ac}^{\ \ b}\,(t - c)_b^{\ ac} + \frac{1}{4}(t - c)_{ac}^{\ \ b}(t - c)^{ac}{}_b +$$

$$- (t' - c')_a (t' - c')^a + 2e_a(t' - c')^a + \frac{1}{2} c_{ab}{}^c (t - c)^{ab}{}_c$$

$$- c_{ab}{}^c (t - c)_c{}^{ab} - c_{ab}{}^{\dot{c}} (t - c)_{\dot{c}}{}^{ab} + conj.\ terms.$$

Most of the tc cross terms cancel. An integration by parts in the action density allows us to replace the fourth term by $-2c'_a (t' - c')^a$; then the expression for eR reduces to

$$\text{(A.3)} \qquad e\left\{\frac{1}{2} c_{ac}{}^b c_b{}^{ac} - \frac{1}{4} c_{ac}{}^b c^{ac}{}_b + \right.$$
$$\left. + c_{ab}{}^{\dot{c}} c_{\dot{c}}{}^{ab} + c'_a c'^a - c_{ab}{}^{\dot{c}} t_{\dot{c}}{}^{ab} + conj.\right\}$$
$$- e\left\{\frac{1}{2} t_{ac}{}^b t_b{}^{ac} - \frac{1}{4} t_{ac}{}^b t^{ac}{}_b + \right.$$
$$\left. + t'_c t'^c + conj.\ terms\right\}.$$

Variation of $\phi_a{}^{bc}$ is tantamount to variation of $t_a{}^{bc}$ and leads to $t_{\dot{a}b}{}^c = 0$. Variation of $\phi_{\dot{a}bc}$ amounts to variation of the symmetric part ot $t_{\dot{a}bc}$ and leads to a constraint on the holonomy coefficients. This is because the bilinear part of R is degenerate. Note that $c_{ab\dot{c}} = t_{ab\dot{c}}$ so that the constraint is covariant.

Constraints on the holonomy coefficients are familiar in superspace formulations of supergravity, but this is the first time that they have turned up in ordinary gravity, as far as we know. The action density (A.3) is unsatisfactory, not because it leads to constraints but because it fails to fix the components ϕ_{abc} of the connection. The covariant way to fix these components is by making the symmetric part of t_{abc} vanish, but we reject the option of doing so by hand. The natural remedy is to add torsion terms. Let us replace R by

$$\text{(A.4)} \qquad R' = R + [p\, t_{ab}{}^{\dot{c}} t_{\dot{c}}{}^{ab} + q\, t^{\dot{a}}{}_{bc}\, t_{\dot{a}}{}^{bc} + r\, t^{\dot{a}}{}_{bc}\, t_{\dot{a}}{}^{cb}$$
$$+ s\, t_{ab}{}^{\dot{c}}\, t^{ab}{}_{\dot{c}} + conj.\ terms].$$

Variation of ϕ_{abc} no longer leads to a constraint on the holonomy coefficients but instead fixes these components,

$$\text{(A.5)} \qquad \phi_{\dot{a}bc} = -\frac{1}{2}(c_{\dot{c}ab} + c_{\dot{c}ba}) + [(1-p)/2(q+r)]\, c_{ab\dot{c}},$$

provided that $q + r \neq 0$. Another way to write this is

$$t_{\dot{a}bc} + t_{\dot{a}cb} = (1-p)/(q+r)\, c_{bc\dot{a}}.$$

Substitution of the variational equations back into R' leads to

$$R' = \frac{1}{2} c_{ac}{}^{b} c_{b}{}^{ac} - \frac{1}{4} c_{ac}{}^{b} c^{ac}{}_{b} + c'_{a} c'^{a} + c_{ab}{}^{\dot{c}} c_{\dot{c}}{}^{ab}$$

$$+ [3(p-1)^2/4(q+r)]\, c_{\dot{a}b}{}^{\dot{c}} c^{ab}{}_{\dot{c}} + \frac{1}{2}(q+r)\, c^{\dot{a}}{}_{bc}(c_{a}{}^{bc} - c_{\dot{a}}{}^{cb}) + conj.$$

Is this theory equivalent to General Relativity? To find out, we express the holonomy coefficients in terms of the Cristoffel symbols. We find that the Einstein-Hilbert action is recovered, provided that the parameters satisfy

$$(A.6) \qquad r - q = 1, \; (q+r)(4s+1) + 3(1-p)^2 = 0.$$

The possibility of a generalization of the minimal coupling corresponds to the generalizations that are possible in the more standard approach.

Turning to supergravity, we note that the expressions (3.19) and (5.5) are formally equal once the constraints $c_{ab}{}^{\gamma} = 0$ are satisfied. The coefficients $\phi_{a}{}^{bc}$, $\phi_{a}{}^{\beta\gamma}$ are related by

$$\phi_{a}{}^{bc} = \frac{1}{2} (\Sigma_{\beta\gamma})^{bc} \phi_{a}{}^{\beta\gamma}, \quad \phi_{a}{}^{\beta\gamma} = (\Sigma^{\beta\gamma})_{bc} \phi_{a}{}^{bc}.$$

We found that the action density eR is unsatisfactory in gravity because variation fails to fix the components $\phi_{\dot{a}bc}$, and the same objection holds in supergravity. Here too we can add torsion terms, but now the requirements of invariance are more severe. The most general bilinear scalar is a combination of $t.t$ and $t'.t'$, where

$$(A.7) \qquad t \cdot t = \eta^{ab}\, t_{bC}{}^{D}\, t_{aD}{}^{C}.$$

Capital indices take 8 values; $t_{a}{}'$ is the supertrace $t_{aB}{}^{B}$.

At this point we run into a problem that has not found a solution until now. The expression (A.7), that must be included in the action density in order that the variational principle fix all the components of the Lorentz connection, introduces the component $t_{a\beta}{}^{c}$ and through them the holonomy coefficients $c_{a\beta}{}^{c}$. This means that the action density now involves the vectorial vierbein e', absent from the curvature scalar and from the formulation of interactions with matter. For this reason we are not enthusiastic about adding torsion terms to the action, and we hope find a more satisfactory way to fix the connection coefficients.

We do not expect to find much application for this formulation of General Relativity, since difficulties are encountered as soon as one attempts to use it in connection with fermions. But the developments of this Section will be useful later, when we turn to supergravity.

Remark. We return to the constraint $c_{ab}{}^{\dot{c}} = 0$ that one obtains when the action density is simply eR. There is one case in which this may be acceptable. Suppose we have a theory in which only the Q_a, but not the $Q_{\dot{a}}$, are used; then there is no need to fix the components $\phi_{\dot{a}}{}^{bc}$ of the connection. The vanishing of $c_{ab}{}^{\dot{c}}$ means that the commutators of two Q_a's can be expressed without bringing in the $Q_{\dot{a}}$'s. The covariant Klein-Gordon operator must be taken to be defined as $Q^a Q_a$ (two terms) and the metric $g^{\mu\nu} = e^{a\mu} e_{a\nu}$ is, formally' degenerate. This theory is a useful model for supergravity. It makes little sense as a classical field theory, but the quantized metric field is not necessarily unreasonable. It would be interesting to attempt a definition of the vierbein field as a quantum field operator without vacuum expectation value, and recover the vacuum expectation value of the metric as a quantum anomaly.

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